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Numerical pricing of financial derivatives using Jain's high-order compact scheme

Nawdha Thakoor*, Yannick Tangman and Muddun Bhuruth

Abstract

Purpose: This paper develops new fast and accurate computational schemes for pricing European and American bond options under generalised Chan-Karoyli-Longstaff-Sanders term structure models.

Methods: We use high-order compact discretisations of the pricing equations and an operator splitting method for American options.

Results: Highly accurate numerical solutions can be computed using relatively coarse grid sizes and numerical solutions exhibiting fourth-order convergence are obtained for bond and bond option prices. The scheme is also stable and efficient for pricing financial problems with time dependent parameters.

Conclusions: The new schemes are efficient alternatives to schemes based on the Crank-Nicolson discretisation for the pricing of interest rate derivatives.

Keywords: Interest rate models, American options, High-order discretisations, Operator splitting methods, Black-Scholes equation

Background

The price of a contingent claim such as an equity or interest rate derivative can often be expressed as the solution of a parabolic equation. For example, under the lognormal diffusion process for stock price evolution of Black-Scholes [1], closed form expressions can be obtained for European options. However, for options with early exercise features, no analytical solutions exist and the pricing has to be carried out numerically. For such problems, the Crank-Nicolson scheme is often the method of choice among practitioners. In a previous work [2], we considered a numerical scheme developed by Jain and his co-authors [3] (henceforth referred as Jain's scheme) for quasilinear parabolic partial differential equations for solving the European pricing problem for equity derivatives. In this present paper, we propose an analysis of Jain's scheme in the framework of three-point schemes discussed by Rigal [4] and we then describe some novel applications to the pricing of interest rate derivatives.

The generalised Chan-Karolyi-Longstaff-Sanders (CKLS) family of term structure models nests two of the most popular interest rate models, namely the Vasicek [5] and the Cox-Ingersoll-Ross (CIR) [6] models. Analytical solutions for bond prices and European bond options exist only for these two models and for other cases, numerical pricing is required. Sorwar, Barone-Adesi and Allegretto [7] proposed a Box scheme for computing prices under CKLS. A faster and more accurate technique using exponential time integration was proposed in [8]. In this work, in contrast to second-order approximations employed in these two papers, we compute numerical solutions for bonds and European bond options which exhibit fourth-order convergence.

We then consider the pricing of American interest rate derivatives. This problem has received considerably much less research attention than the American stock option problem. For the CIR model, a quasi-analytical formula expressing the American option price as the sum of the corresponding European option price and an early exercise premium was derived by Chesney, Elliott and Gibson [9] but no price approximations were given in their paper since this formula is not very easy to compute. The simplified binomial approach of Tian [10] converges only for certain combination of parameters and it is well-known that binomial processes converge slowly. Allegretto, Lin

*Correspondence: nawdha.thakoor@umail.uom.ac.mu
Department of Mathematics, University of Mauritius, Reduit, Mauritius

and Yang [11] proposed finite volume and finite element schemes using second-order approximations, but they considered only the CIR model. The high-order scheme proposed in this paper is shown to accurately compute American option prices on both zero-coupon and coupon-bearing bonds. Therefore the scheme is a useful alternative to the popular Crank-Nicolson scheme and tree-based methods for pricing interest rate derivatives.

This work is organised as follows. We first describe the partial differential equations approach to the pricing of financial derivatives and give the analytical solutions which are known in the literature. We then consider the numerical pricing of options under the Black-Scholes model and provide an analysis of Jain's scheme for solving the European option problem. New schemes for the pricing of European and American interest rate derivatives are developed next and several numerical examples are described to indicate the accuracy and efficiency of the scheme for pricing various derivatives including financial problems with time-dependent parameters. A final section summarises our work.

The partial differential equations framework

This section describes the partial differential equations approach to the pricing of financial derivatives. We first consider the valuation problem associated with equity derivatives, that is, options on stocks.

The Black-Scholes model

We consider a financial market consisting of a risky asset with price process $\{S_t\}_{t\geq 0}$ and constant volatility $\sigma > 0$ in a risk neutral economy with fixed rate of return $r > 0$. Under the risk neutral measure $\mathbb{Q}$, the dynamics of the Black-Scholes model is given by

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sigma dW_t,$$

where δ denotes the continuous dividend yield and W_t is standard $\mathbb{Q}$-Brownian motion.

A European call option with strike price K and maturity date T on the risky asset gives the right to the option holder to buy the asset or not at maturity date. Therefore the payoff at time T of the call option is $g(S_T) = \max(S_T - K, 0) = (S_T - K)^+$. The fundamental option pricing problem is to calculate the fair price that the option holder must pay to acquire this right. Since the option is a tradeable contract, it has a time t price $V(S,t)$ which computed using the martingale approach is the discounted expected payoff under the risk-neutral measure $\mathbb{Q}$ given by

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[g(S_T)|S_t = S\right].$$

Using the Feynman-Kac Theorem, it can be shown that $V(S,t)$ is the solution of the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0, \quad S > 0, \quad 0 \le t \le T, \tag{1}$$

with terminal condition $V(S,T) = g(S_T)$ for a call option. An analytical solution for (1) exists and is given by

$$V(S,t) = Se^{-\delta(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

where Φ is the distribution function of the standard normal distribution $Z \sim N(0,1)$ and

$$d_2 = \frac{\log(S/K) + \left((r-\delta) - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_1 = d_2 + \sigma\sqrt{T-t}.$$

The CKLS stochastic interest rate model

In this section, we assume that the spot rate $r(t)$ at time t is stochastic. Chan, Karoyli, Longstaff and Sanders [12] developed a general framework for stochastic interest rate models which nests two well-known term structure models. The framework which we denote by CKLS assumes that the spot rate $r(t)$ under the risk neutral measure $\mathbb{Q}$ is governed by the stochastic differential equation

$$dr(t) = \kappa(\theta - r(t))dt + \sigma r(t)^{\gamma} dW_t, \tag{2}$$

where κ, θ, σ and γ are positive constants. The parameter κ is the rate of reversion about the long run mean θ, σ is the volatility of the diffusion and γ is the key parameter used for nesting the different models.

Under the risk neutral measure $\mathbb{Q}$, the market price of risk defined as the excess expected instantaneous return above the riskless rate divided by the instantaneous standard deviation of return is zero and using no-arbitrage arguments, it can be shown that the price $V(r,t)$ at time t of a financial contract with terminal payoff $g(r)$ at time T is the solution of

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 V}{\partial r^2} + \kappa(\theta - r)\frac{\partial V}{\partial r} - rV = 0, \qquad V(r,T) = g(r). \tag{3}$$

For a zero-coupon bond with maturity T and face value of one dollar (such a bond is also called a unit discount bond), we denote the price at time t by $P(r,t,T)$ and we need to solve (3) subject to the terminal payoff $g(r) = 1$. For a European call option on the zero-coupon bond with strike K and maturity $T_o < T$, let $C(r,t,T_o,T,K)$ denote the European call option price. Then we need to solve (3) subject to the terminal payoff $g(r) = (P(r,T_o,T) - K)^+$.

Analytical solutions

Analytical solutions to the problem (3) exist only for the cases $\gamma = 0$ which corresponds to the Vasicek model [5] and for the Cox-Ingersoll-Ross model [6] with $\gamma = 1/2$. Under these two term structure models, the price $P(r, t, T)$ of the discount bond has the form

$$P(r, t, T) = A(t, T)e^{-r(t)B(t,T)}. \tag{4}$$

For the Vasicek model, expressions for $B(t, T)$ and $\ln A(t, T)$ are given by

$$B(t, T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right),$$
$$\ln A(t, T) = \left(\theta - \frac{\sigma^2}{2\kappa^2}\right)(B(t, T) - T + t) - \frac{\sigma^2}{4\kappa}B^2(t, T).$$

The European call option under the Vasicek model has price

$$C(r, t, T_o, T, K) = P(r, t, T)\,\Phi(\hbar) - KP(r, t, T_o)\,\Phi(\hbar - \sigma_p),$$

where

$$\sigma_p = \sqrt{\frac{1 - e^{-2\kappa(T-t)}}{2\kappa}}B(T_o, T),$$
$$\hbar = \frac{1}{\sigma_p}\ln\left(\frac{P(r, t, T)}{KP(r, t, T_o)}\right) + \frac{1}{2}\sigma_p.$$

For the Cox-Ingersoll-Ross model, letting $\eta = \sqrt{\kappa^2 + 2\sigma^2}$, the functions $A(t, T)$ and $B(t, T)$ in (4) are given by

$$A(t, T) = \left(\frac{2\eta e^{(\eta+\kappa)(T-t)/2}}{2\eta + (\eta+\kappa)\left(e^{\eta(T-t)} - 1\right)}\right)^{2\kappa\theta/\sigma^2},$$
$$B(t, T) = \frac{2\left(e^{\eta(T-t)} - 1\right)}{2\eta + (\eta+\kappa)\left(e^{\eta(T-t)} - 1\right)}.$$

The expression for a European call option under the Cox-Ingersoll-Ross model is given by

$$V_c(r, t, T_o, T, K) = P(r, t, T)\,\chi^2(a, \hat{b}, c) - KP(r, t, T_o)\,\chi^2(d, \hat{b}, y), \tag{5}$$

where

$$\begin{aligned} a &= 2\bar{r}\left(\phi + \psi + B(T_o, T)\right), & \hat{b} &= \tfrac{4\kappa\theta}{\sigma^2}, \\ c &= \tfrac{2\phi^2 r e^{\eta(T-t)}}{\phi+\psi+B(T_o,T)}, & d &= 2\bar{r}(\phi+\psi), \\ y &= \tfrac{2\phi^2 r e^{\eta(T-t)}}{\phi+\psi}, & \bar{r} &= \tfrac{1}{B(T_o,T)}\ln\left(\tfrac{A(T_o,T)}{K}\right), \\ \phi &= \tfrac{2\eta}{\sigma^2\left(e^{\eta(T-t)}-1\right)}, & \psi &= (\kappa+\eta)/\sigma^2, \end{aligned}$$

and where $\chi^2(z, \nu, l)$ denotes the cumulative distribution function of a noncentral Chi-square random variable with noncentrality parameter l and ν degrees of freedom. The price V_p of a European put can be obtained using the put-call parity for bond options given by

$$V_p(r, t, T_o, T, K) = V_c(r, t, T_o, T, K) - P(r, t, T) + KP(r, t, T_o).$$

One difficulty which may arise with the evaluation of the explicit formulas in (5) is that the computation of the Chi-square distribution can be slow for some parameter values. For equity option pricing under the constant elasticity of variance (CEV) model which also involves the non-central Chi-square distribution, Wong and Zhao [13] showed that a finite difference method can be a faster alternative to the evaluation of the European option formula when the elasticity factor in the CEV model is close to one, volatility is low or time to maturity is small.

Options on coupon bonds

We consider a coupon bond with face value $\check{f}$ and maturity T and we assume that predetermined payments of $\check{a}_i$ at specified time periods T_i for $1 \le i \le \tilde{N}$ where $T_0 = 0$ and $T_{\tilde{N}} = T$ are paid to the bond holder. Then the time zero price $P_c(r, 0, T)$ of the coupon bond is given by

$$P_c(r, 0, T) = \sum_{i=1}^{\tilde{N}} \check{a}_i P(r, 0, T_i). \tag{6}$$

Closed form expressions for European options on coupon-bearing bonds have been derived by Jamshidian [14] for the Vasicek model and by Longstaff [15] for the CIR model. In the numerical example, we have considered the pricing of European and American options under the CIR model. We therefore give below the closed form formula for a European option with maturity T_o on a coupon bond with maturity T. For $i = 1, 2, \ldots, \hat{N}$, let $T_i > T_o$ denote the payment dates and let $T_{b_i} = T_i - T_o$. The European option with strike price K has payoff $\left(\sum_{i=1}^{\hat{N}} \check{a}_i P(r, 0, T_i) - K\right)^+$. Denoting by r^* the value of the interest rate r which solves the equation

$$\sum_{i=1}^{\hat{N}} \check{a}_i P(r, 0, T_{b_i}) = K,$$

the value $V_c(r, T_o, T, K)$ of the European option at $t = 0$ is given by

$$V_c(r, T_o, T, K) = \sum_{i=1}^{\hat{N}} \check{a}_i P(r, 0, T_o + T_{b_i})\chi^2\left(\varphi_i, \hat{b}, \vartheta_i\right) - KP(r, 0, T_o)\chi^2\left(\varphi_0, \hat{b}, \vartheta_0\right), \tag{7}$$

where

$$\varphi_0 = \frac{4\eta\left(1+\varrho e^{\eta T_o}\right)r^*}{\sigma^2(1+\varrho)\left(e^{\eta T_o}-1\right)}, \qquad \varphi_i = \frac{4\eta\left(1+\varrho e^{\eta(T_o+T_{b_i})}\right)r^*}{\sigma^2\left(1+\varrho e^{\eta T_{b_i}}\right)\left(e^{\eta T_o}-1\right)},$$

$$\varrho = \frac{\eta+\kappa}{\eta-\kappa}, \qquad \vartheta_0 = \frac{4\eta e^{\eta T_o}(1+\varrho)r}{\sigma^2\left(1+\varrho e^{\eta T_o}\right)\left(e^{\eta T_o}-1\right)},$$

$$\vartheta_i = \frac{4\eta e^{\eta T_o}\left(1+\varrho e^{\eta T_{b_i}}\right)r}{\sigma^2\left(1+\varrho e^{\eta(T_o+T_{b_i})}\right)\left(e^{\eta T_o}-1\right)}, \quad \hat{b} = \frac{4\kappa\theta}{\sigma^2}.$$

American options

The availability of the analytical solutions for the Vasicek and Cox-Ingersoll-Ross models provides us with a framework for assessing our new numerical scheme and then proceed to the pricing of American options for which no analytical solutions exist. For an American put option with strike K and maturity $T_o < T$ on a zero-coupon bond with maturity T, although the underlying asset is the bond, the independent variable is the stochastic interest rate r and there exists an unknown optimal exercise interest rate $r_f(t)$ at each time t for which the exercise of the option becomes optimal.

One approach to the pricing of American options is to relate the American option price to an optimal stopping problem [16]. Using this approach, it is shown in [9] that the American put option price can be written as a sum of the corresponding European price and an early exercise premium e_p given by

$$e_p = \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{T_o} \exp\left(-\int_t^s r(w)dw\right) r(s)\mathbf{1}_{r(s)\geq r_f(s)}\, ds\right],$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes expectation at time t under the risk-neutral measure $\mathbb{Q}$ and $\mathbf{1}_{\mathfrak{F}}$ denotes the indicator function for the set $\mathfrak{F}$. We show below that the computation of this early exercise premium is difficult.

Quasi-analytical solution for Cox-Ingersoll-Ross

The above early exercise premium term can be simplified for the case of the CIR model using the properties of Bessel bridges [9]. Define the process ρ by $\rho\left(\frac{\sigma^2 w}{4}\right) = r(w)$ and let $j = -2\kappa/\sigma^2$, $r_1 = \rho\left(\frac{\sigma^2 t}{4}\right)$, $r_2 = \rho\left(\frac{\sigma^2 s}{4}\right)$, $\varsigma = \sqrt{j^2 + 8/\sigma^2}$ and $\zeta = \varsigma\sigma^2/4$. Also let

$$\begin{aligned}\tilde{m}(r(s), s) =& \frac{\zeta(s-t)}{\sinh \zeta(s-t)} \exp\left(j(r_2 - r_1) - \kappa\theta(s-t)\right)\\ &\times \exp\left[\frac{2(r_1+r_2)}{\sigma^2(s-t)}\left(1-\frac{\sigma^2}{4}\varsigma(s-t)\coth \zeta(s-t)\right)\right]\\ &\times \frac{I_{\frac{\hat{n}}{2}-1}\left(\varsigma\sqrt{r_1 r_2}/\sinh \zeta(s-t)\right)}{I_{\frac{\hat{n}}{2}-1}\left(\varsigma\sqrt{r_1 r_2}/\left(\frac{\sigma^2}{4}(s-t)\right)\right)},\end{aligned}$$

where I_υ denotes a modified Bessel function of order υ. Then, the early exercise premium is given by

$$e_p = K\int_t^{T_o}\int_{r_f(s)}^{\infty} r(s)\tilde{m}(r(s), s)f(r(s), s)dr(s)ds, \qquad (8)$$

where $f(r(s), s)$ is the density function of the interest rate r at time s conditional on its value at time t given by

$$f(r(s), s) = ce^{-x-y}(y/x)^{q/2}I_q(2\sqrt{xy}),$$

with

$$c = \frac{2\kappa}{\sigma^2\left(1-e^{-\kappa(s-t)}\right)},\ x = cr(t)e^{-\kappa(s-t)},\ \nu = cr(s),\ q = \frac{2\kappa\theta}{\sigma^2} - 1.$$

As pointed out in [9], the early exercise premium formula (8) is complex and is less tractable than the early exercise premium formulas in the case of the Black-Scholes model for American stock options. We therefore need to develop alternative solution methods for the pricing of American interest rate claims. In the next section, we describe the free-boundary and the linear complementarity formulations of the American put.

Free-boundary and linear complementarity problems

Let $\hat{g}(r, t) = (K - P(r, t, T_o))^+$ and consider the differential operator

$$\mathcal{L}_r = \frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2}{\partial r^2} + \kappa(\theta - r)\frac{\partial}{\partial r} - r.$$

Under the free-boundary formulation, the price $V(r, t)$ of the American put at time t given the spot rate $r(t) = r$ is the solution of the problem

$$\begin{aligned}&\frac{\partial V}{\partial t} + \mathcal{L}_r V = 0, \quad 0 < r < r_f(t), \quad 0 \leq t < T_o,\\ &V(r, t) > \hat{g}(r, t), \quad 0 < r < r_f(t), \quad 0 \leq t < T_o,\\ &V(r_f(t), t) = K - P\left(r_f(t), t, T_o\right), \quad 0 \leq t < T_o,\\ &\frac{\partial V}{\partial r}\left(r_f(t), t\right) = \frac{\partial \hat{g}}{\partial r}\left(r_f(t), t\right), \quad 0 \leq t < T_o,\\ &V(0, t) = \hat{g}(0, t), \quad 0 \leq t < T_o,\\ &V(r, T_o) = \hat{g}(r, T_o), \quad r \geq 0.\end{aligned} \qquad (9)$$

The explicit dependence of (9) on the free-boundary means that an accurate location of $r_f(t)$ is required to produce an efficient pricing technique. In [11], a finite element scheme under the CIR model was derived by using a transformation which removes the degenerate factor in the highest derivative term of the free-boundary formulation. Our scheme is different in two aspects. First we work with the linear complementarity formulation of the American put which does not require the explicit computation of the free-boundary and second we use a high-order finite difference scheme in an operator splitting framework. The problem we solve is given by

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \mathcal{L}_r V \le 0, \quad r > 0, \quad 0 \le t \le T_o,\\
&V(r, T_o) = \hat{g}(r, T_o), \quad r > 0,\\
&V(r, t) \ge \hat{g}(r, t), \quad r > 0, \quad 0 \le t \le T_o, \qquad (10)\\
&\left(\frac{\partial V}{\partial t} + \mathcal{L}_r V\right) \cdot \big(V(r, t) - \hat{g}(r, t)\big) = 0, \quad r > 0, \quad 0 \le t \le T_o.
\end{aligned}$$

A similar problem needs to be solved for the American option on a coupon-bond and more details are given later in the paper.

Methods

Having described the problems considered in this paper, next we describe the numerical discretisations of the different problems. We first consider a high-order compact discretisation of a transformed Black-Scholes equation and we then describe some new numerical schemes for pricing interest rate derivatives.

A numerical scheme for Black-Scholes equation

We consider the substitutions given by $S = Ke^x$, $\tau = \sigma^2(T - t)/2$, $p_\delta = 2(r - \delta)/\sigma^2$, $T' = \sigma^2 T/2$ and $p = 2r/\sigma^2$ to transform the Black-Scholes pde (1) to a constant-coefficient problem given by

$$\begin{aligned}
&\frac{\partial V}{\partial \tau} = \frac{\partial^2 V}{\partial x^2} + (p_\delta - 1)\frac{\partial V}{\partial x} - pV, \ -\infty < x < +\infty,\\
&\qquad 0 < \tau \le T', \qquad (11)\\
&V(x, 0) = K\left(e^x - 1\right)^+, \quad -\infty < x < +\infty,\\
&V(x, \tau) = 0, \quad x \to -\infty, \quad 0 < \tau \le T',\\
&V(x, \tau) = K(e^{x - 2\delta\tau/\sigma^2} - e^{-p\tau}), \quad x \to +\infty.
\end{aligned}$$

A final substitution $u(x, \tau) = e^{p\tau} V(x, \tau)$ reduces (11) to

$$\begin{aligned}
&\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (p_\delta - 1)\frac{\partial u}{\partial x}, \quad -\infty < x < +\infty,\\
&\qquad 0 < \tau \le T', \qquad (12)\\
&u(x, 0) = K(e^x - 1)^+, \quad -\infty < x < +\infty,\\
&u(x, \tau) = 0, \quad x \to -\infty, \quad 0 < \tau \le T',\\
&u(x, \tau) = K\left(e^{x + (p - 2\delta/\sigma^2)\tau} - 1\right), \quad x \to +\infty.
\end{aligned}$$

To obtain the numerical scheme, we first localise (12) to a finite domain $\Omega = (x_{\min}, x_{\max}) \times [0, T']$. Then considering mesh-spacings of $h = (x_{\max} - x_{\min})/M$ in the x-direction and $k = T'/N$ in the time direction, we have the uniform mesh of grid points

$$\begin{aligned}
\Omega_\Delta = \{&(x_m, \tau_n) \in \Omega, x_m = x_{\min} + mh, 0 \le m \le M,\\
&\tau_n = nk, 0 \le k \le N\}
\end{aligned}$$

on Ω and we let u_m^n denote the approximate value of $u(x_m, t_n)$. Let $b = 1 - p_\delta$ and write (12) in the form

$$\frac{\partial^2 u}{\partial x^2} = b\frac{\partial u}{\partial x} + \frac{\partial u}{\partial \tau} = f\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial \tau}\right). \qquad (13)$$

Jain's scheme for (30) is derived from a Numerov discretisation of (30) in the form

$$-\frac{1}{h^2}\delta_x^2 u_m^{n+\frac{1}{2}} + \frac{1}{12}\left(f_{m+1}^{n+\frac{1}{2}} + 10 f_m^{n+\frac{1}{2}} + f_{m-1}^{n+\frac{1}{2}}\right) = 0, \qquad (14)$$

using the approximations

$$\begin{aligned}
\delta_x^2 u_m^{n+\frac{1}{2}} &\approx u_{m-1}^{n+\frac{1}{2}} - 2u_m^{n+\frac{1}{2}} + u_{m+1}^{n+\frac{1}{2}},\\
u_{m+q}^{n+\frac{1}{2}} &\approx \frac{1}{2}\left(u_{m+q}^{n+1} + u_{m+q}^{n}\right),\\
&\qquad \text{for } q = 0, \pm 1,\\
f_{m+q}^{n+\frac{1}{2}} &\approx \bar{f}_{m+q}^{n+\frac{1}{2}} = b(u_x)_{m+q}^{n+\frac{1}{2}} + \frac{u_{m+q}^{n+1} - u_{m+q}^{n}}{k},\\
&\qquad \text{for } q = 0, \pm 1,\\
(u_x)_{m\pm1}^{n+\frac{1}{2}} &\approx \pm\frac{1}{2h}\left(3u_{m\pm1}^{n+\frac{1}{2}} - 4u_m^{n+\frac{1}{2}} + u_{m\mp1}^{n+\frac{1}{2}}\right), \qquad (15)\\
(u_x)_m^{n+\frac{1}{2}} &\approx \frac{1}{2h}\left(u_{m+1}^{n+\frac{1}{2}} - u_{m-1}^{n+\frac{1}{2}}\right) - \frac{h}{20}\left(\bar{f}_{m+1}^{n+\frac{1}{2}} - \bar{f}_{m-1}^{n+\frac{1}{2}}\right).
\end{aligned}$$

Letting

$$\begin{aligned}
&\beta_{-1} = \frac{1}{12} - \frac{k}{2h^2} - \frac{b^2 k}{24}, \quad \beta_1 = \frac{1}{12} + \frac{k}{2h^2} + \frac{b^2 k}{24},\\
&\gamma_{-1} = b\left(\frac{k}{4h} - \frac{h}{24}\right), \gamma_1 = b\left(\frac{k}{4h} + \frac{h}{24}\right),
\end{aligned}$$

gives the scheme

$$\begin{aligned}
(\beta_{-1} - \gamma_{-1})u_{m-1}^{n+1} &+ (1 - 2\beta_{-1})u_m^{n+1} + (\beta_{-1} + \gamma_{-1})u_{m+1}^{n+1}\\
&= (\beta_1 + \gamma_1)u_{m-1}^{n} + (1 - 2\beta_1)u_m^{n}\\
&\quad + (\beta_1 - \gamma_1)u_{m+1}^{n}. \qquad (16)
\end{aligned}$$

Analysis of numerical scheme

The stability analysis of the scheme in (16) has been carried out in [3]. Here we present a different analysis in the framework of general two-level three-point schemes of the form

$$(1+C)\,D_t u_m^n = \left(\frac{1}{2}+A_1\right) D_+D_-u_m^n + \left(\frac{1}{2}+A_2\right) D_+D_-u_m^{n+1} - b\left(\frac{1}{2}+B_1\right) D_0 u_m^n - b\left(\frac{1}{2}+B_2\right) u_m^{n+1}, \tag{17}$$

with the difference operators given by

$$D_t u_m^n = \frac{u_m^{n+1}-u_m^n}{k}, \qquad D_0 u_m^n = \frac{u_{m+1}^n - u_{m-1}^n}{2h},$$
$$D_+ u_m^n = \frac{u_{m+1}^n - u_m^n}{h}, \qquad D_- u_m^n = \frac{u_m^n - u_{m-1}^n}{h}.$$

It is then easy to see that (16) corresponds to (17) for the values

$$C=0, \qquad A_1 = \tfrac{b^2h^2}{24}+\tfrac{h^2}{12k}, \qquad A_2 = \tfrac{b^2h^2}{24}-\tfrac{h^2}{12k},$$
$$B_1 = \tfrac{h^2}{12k}, \qquad B_2 = -\tfrac{h^2}{12k}. \tag{18}$$

The local truncation error T_m^n at the grid point (x_m, t_n) is given by

$$\begin{aligned} T_m^n =& \left(\frac{b^3k^2}{12}\right)\partial_x^3 u + \left[\frac{b^2h^4}{144}-\frac{b^2k^2}{4}-\frac{1}{144}b^4h^2k^2-\frac{b^4k^3}{24}\right]\partial_x^4 u \\ &+ \left[\frac{bk^2}{4}+\frac{b^3h^2k^2}{24}+h^4\left(\frac{b^3k}{288}-\frac{b}{80}\right)\right]\partial_x^5 u \\ &+ \left[-\frac{k^2}{4}+h^2\left(\frac{k}{12}-\frac{b^2k^2}{16}\right) \right. \\ &\left. +h^4\left(\frac{1}{240}-\frac{b^2k}{288}\right)-\frac{b^2h^6}{4320}\right]\partial_x^6 u + \text{HOD} \end{aligned}$$

where HOD stands for higher order derivatives. Expressing the local truncation error in the form

$$\begin{aligned} T_m^n =& \left(\frac{b^3k^2}{12}\right)\partial_x^3 u + \left[\frac{b^2h^4}{144}-\frac{b^2k^2}{4}\right]\partial_x^4 u \\ &+ \left[\frac{bk^2}{4}-\frac{bh^4}{80}\right]\partial_x^5 u + \left[\frac{kh^2}{12}-\frac{k^2}{4}+\frac{h^4}{240}\right]\partial_x^6 u \\ &+ \mathcal{O}(k^3+h^2k^2+h^6). \end{aligned}$$

we see that the choice $k=\mu h^2$ for some appropriate constant μ gives a truncation error $T_m^n = \mathcal{O}(h^4)$ for smooth problems. One difficulty is that the payoff function g has a discontinuous first derivative at the strike price K which will prevent the scheme from achieving the expected fourth-order convergence rate. In [2] we used a grid-stretching transformation to recover the expected convergence rate. The fourth-order numerical results that we present in this paper are based on two different aspects. First we solve a convection-diffusion problem instead of the heat equation and second we employ a local mesh refinement technique [17] instead of a more complicated coordinate transformation technique.

Stability and non-oscillatory properties

Rigal [4] has shown that the numerical scheme (17) is consistent with the differential problem (30) if $C = B_1 + B_2$ and that the scheme is forward diffusive if

$$1+C>0, \quad 1+A_1+A_2>0. \tag{19}$$

Considering the coefficients C, A_1 and A_2 given in (18) for the scheme (16) we see that both the consistency and forward diffusivity conditions are satisfied. In addition the numerical scheme (17) is stable if

$$kb^2(B_1-B_2) \le 2(1+A_1+A_2), \tag{20}$$
$$2k(A_1-A_2) \le h^2(1+C),$$

and is non-oscillatory if

$$2(1+A_1+A_2) \le bh(1+C). \tag{21}$$

Using the coefficients given in (18) for numerical scheme in (16), it is easy to verify that the stability and the non-oscillatory conditions are always satisfied. We thus conclude that Jain's scheme for solving the Black-Scholes equation has all the favourable properties of a good numerical scheme.

Numerical scheme for CKLS

In a previous work [8] we developed a second-order numerical scheme for pricing bond and European bond options under CKLS. The proposed method in this paper is different from the scheme described in [8] in several aspects. First, the numerical scheme is a high-order scheme for pricing bond and bond options. Second, our scheme is a fully discrete scheme whereas the scheme employed in [8] is a semi-discrete one with the time integration carried out using an exponential integrator. Thirdly, we also price American and Bermudan options on coupon bonds under the CIR model and we also solve financial problems which are described by equations with time-dependent coefficients.

We describe our numerical scheme for pricing a unit discount bond. Using the substitution $\tilde{\tau} = T - t$, (3) is transformed to a forward problem where the discount bond price $P(r, \tilde{\tau}, T)$ at time $\tilde{\tau}$ is the solution of

$$\frac{\partial P}{\partial \tilde{\tau}} = \frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 P}{\partial r^2} + \kappa(\theta - r)\frac{\partial P}{\partial r} - rP, \tag{22}$$

with initial condition $P(r, 0, T) \equiv 1$.

Truncating the r-domain $0 \leq r < \infty$ to $\Omega_r = (r_{\text{rmin}}, r_{\text{rmax}})$, we consider a uniform grid with spacing $h = (r_{\text{rmax}} - r_{\text{rmin}})/M$ in the r-direction and a spacing of $k = T/N$ in the time direction. Then our mesh Ω_Δ superimposed on $\Omega = \Omega_r \times [0, T]$ is given by

$$\Omega_\Delta = \{(r_m, \tilde{\tau}_n) \in \Omega, r_m = r_{\min} + mh, 0 \leq m \leq M, \\ \tilde{\tau}_n = nk, 0 \leq k \leq N\}.$$

Expressing (22) in the form

$$\begin{aligned} \frac{\partial^2 P}{\partial r^2} &= \frac{2}{\sigma^2 r^{2\gamma}}\left(\frac{\partial P}{\partial \tilde{\tau}} - \kappa(\theta - r)\frac{\partial P}{\partial r} + rP\right) \\ &= f\left(r, P, \frac{\partial P}{\partial \tilde{\tau}}, \frac{\partial P}{\partial r}\right), \end{aligned} \tag{23}$$

and writing the approximate value of the bond price at the grid point (r_m, t_n) by P_m^n, Jain's scheme for (23) is obtained from (14) and the approximations of the derivative terms given in (15) where now

$$f_{m+q}^{n+\frac{1}{2}} \approx \bar{f}_{m+q}^{n+\frac{1}{2}} = f\left(r_{m+q}, P_{m+q}^{n+\frac{1}{2}}, \frac{P_{m+q}^{n+1} - P_{m+q}^n}{k}, (P_r)_{m+q}^{n+\frac{1}{2}}\right), \\ q = -1, 0, 1. \tag{24}$$

Denoting $\xi_{m+q} = -\kappa(\theta - r_{m+q})$ for $q = 0, \pm 1$ gives the fully discrete scheme

$$b_{m-1}P_{m-1}^{n+1} + b_m P_m^{n+1} + b_{m+1}P_{m+1}^{n+1} = c_{m-1}P_{m-1}^n + c_m P_m^n \\ + c_{m+1}P_{m+1}^n, \tag{25}$$

If we let $P^n = \left(P_0^n, P_2^n, \ldots, P_M^n\right)^T$ denote the vector of bond prices, then the numerical scheme (25) can be written in matrix form as

$$AP^{n+1} = BP^n, \quad n \geq 0, \tag{26}$$

where $A = \text{tridiag}[\, b_{m-1}, b_m, b_{m+1}]$ and $B = \text{tridiag}\,[\, c_{m-1}, c_m, c_{m+1}]$ subject to the initial condition $P^0 = \mathbf{1}$ where $\mathbf{1}$ is a vector of ones.

Both bond prices P and bond option prices V satisfy (22) with appropriate boundary conditions. We now describe the solution method for bond options. For a European call option with maturity T_o and strike price K on a discount bond with maturity T, we solve (26) for $V(r, \tau^*, T_0)$ using the initial condition $V(r, 0, T_o) = (P(r, T_o, T) - K)^+$ where $V(r, \tau^*, T_o)$ satisfies (22) in the time variable $\tau^* = T_o - t$.

The price of a bond with face value $\check{f}$ which makes coupon payments of amount $\check{a}$ at regular intervals (annually or semi-annually) is obtained by solving (26) with initial condition $P(r, 0, T) = \check{f} + \check{a}$. We solve for P^{n+1} at each time level by computing $P(r, \tau_{n+1}^*, T)$ and if the time level $n+1$ corresponds to a coupon payment date, we add the coupon, that is, $P(r, \tau_{n+1}^*, T) = P(r, \tau_{n+1}^*, T) + \check{a}$. For a European call option with maturity T_o on a coupon bond with maturity T with initial payoff function

$$V(r, 0, T_o) = (P(r, T_o, T) - K)^+. \tag{27}$$

we solve (26) using the initial condition (27) and, at each time level, we solve for V^{n+1}, that is, we compute $V(r, \tau_{n+1}^*, T_o)$ using $V(r, \tau_n^*, T_o)$ as initial condition. The coupon amount $\check{a}$ is then added if the time level $n + 1$ corresponds to a coupon payment date. For

where

$$b_{m\pm1} = r_{m-1}^{-2\gamma} r_m^{-2\gamma} r_{m+1}^{-2\gamma}\left(\sigma^2 r_m^{2\gamma}\left[\mp hk\xi_{m\mp1} r_{m\pm1}^{2\gamma} + r_{m\mp1}^{2\gamma}\left(4h^2 \pm 3hk\xi_{m\pm1} + 2h^2 k r_{m\pm1} - 12k\sigma^2 r_{m\pm1}^{2\gamma}\right)\right]\right. \\ \left. \mp h\xi_m\left(\pm hk\xi_{m\mp1} r_{m\pm1}^{2\gamma} + r_{m\mp1}^{2\gamma}\left[4h^2 \pm 3hk\xi_{m\pm1} + 2h^2 k r_{m\pm1} - 10k\sigma^2 r_{m\pm1}^{2\gamma}\right]\right)\right),$$

$$c_{m\pm1} = r_{m-1}^{-2\gamma} r_m^{-2\gamma} r_{m+1}^{-2\gamma}\left(\sigma^2 r_m^{2\gamma}\left[\pm hk\xi_{m\mp1} r_{m\pm1}^{2\gamma} + r_{m\mp1}^{2\gamma}\left(4h^2 \mp 3hk\xi_{m\pm1} - 2h^2 k r_{m\pm1} + 12k\sigma^2 r_{m\pm1}^{2\gamma}\right)\right]\right. \\ \left. + h\xi_m\left(hk\xi_{m\mp1} r_{m\pm1}^{2\gamma} + r_{m\mp1}^{2\gamma}\left[\mp 4h^2 + 3hk\xi_{m\pm1} \pm 2h^2 k r_{m\pm1} \mp 10k\sigma^2 r_{m\pm1}^{2\gamma}\right]\right)\right),$$

$$b_m = 4r_m^{-2\gamma}\left(hk\xi_{m-1} r_{m-1}^{-2\gamma}\left(h\xi_m + \sigma^2 r_m^{2\gamma}\right)\right. \\ \left. + r_{m+1}^{-2\gamma}\left[h^2 k\xi_m\xi_{m+1} + \sigma^2\left(-hk\xi_{m+1} r_m^{2\gamma} + r_{m+1}^{2\gamma}\left[10h^2 + 5h^2 k r_m + 6k\sigma^2 r_m^{2\gamma}\right]\right)\right]\right),$$

$$c_m = 4r_m^{-2\gamma}\left(hk\xi_{m+1} r_{m+1}^{-2\gamma}\left(-h\xi_m + \sigma^2 r_m^{2\gamma}\right)\right. \\ \left. + r_{m-1}^{-2\gamma}\left[-h^2 k\xi_m\xi_{m-1} - hk\sigma^2\xi_{m-1} r_m^{2\gamma}\right] + \sigma^2\left[10h^2 - 5h^2 k r_m - 6k\sigma^2 r_m^{2\gamma}\right]\right).$$

a Bermudan call option on a coupon bond where the option can only be exercised at some specified dates, we solve (26) using (27) as the initial payoff function. We then find $V(r, \tau^*_{n+1}, T_o)$ at each time step and if the time level $n+1$ corresponds to a coupon date, we add the coupon, that is, $V(r, \tau^*_{n+1}, T_o) = V(r, \tau^*_{n+1}, T_o) + \check{a}$. Also if the time level $n+1$ is an exercise date, the value of the option becomes $V(r, \tau^*_{n+1}, T_o) = \max\left(V(r, \tau^*_{n+1}, T_o), (P(r, \tau^*_{n+1}, T) - K)^+\right)$.

Operator splitting method for American bond options

Let $\tau^* = T_o - t$ and let the transformed payoff function be given by $\tilde{g}(r, \tau^*) = (K - P(r, \tau^*, T_o))^+$. Using the auxiliary variable $\lambda \geq 0$, the linear complementarity problem (10) can be written in the form

$$\begin{aligned} \frac{\partial V}{\partial \tau^*} - \mathcal{L}_r V - \lambda = 0, \quad \lambda \geq 0, \quad r > 0, 0 \leq \tau^* \leq T_o \\ V(r, 0) = \tilde{g}(r, 0), \quad r > 0, \\ V(r, \tau^*) \geq \tilde{g}(r, \tau^*), \quad 0 \leq \tau^* < T_o, \\ \left[\frac{\partial V}{\partial \tau^*} - \mathcal{L}_r V\right] \cdot \left[V(r, \tau^*) - \tilde{g}(r, \tau^*)\right] = 0, \\ r > 0, 0 \leq \tau^* \leq T_o. \end{aligned} \tag{28}$$

To solve the linear complementarity problem (28), we carry out a discretisation of $\frac{\partial V}{\partial \tau^*} - \mathcal{L}_r V = 0$ similar to the discretisation carried out for (23) using the approximations in (24). We then solve for the pair $\left(V^{n+1}, \lambda^{n+1}\right)$ from

$$AV^{n+1} - BV^n - \lambda^{n+1} = 0, \quad \text{for } 0 \leq n \leq N-1.$$

Ikonen and Toivanen [18] carried out the computation in two fractional steps. The first step involves solving

$$A\tilde{V}^{n+1} - BV^n - \lambda^n = 0, \quad 0 \leq n \leq N-1,$$

for $\tilde{V}^{n+1}$ using LU-decomposition. The second fractional time step computes V^{n+1} and λ^{n+1} using the two-step formula

$$\begin{aligned} V^{n+1} &= \max\left(\left(K - P^{n+1}\right)^+, \tilde{V}^{n+1} - \lambda^n\right), \\ \lambda^{n+1} &= \lambda^n - \left(\tilde{V}^{n+1} - V^{n+1}\right), \end{aligned}$$

and $\lambda^0 = \mathbf{0}$ where $\mathbf{0}$ is a vector of zeros.

Financial problem with time-dependent coefficients

We now consider two problems where the pricing pdes have time-dependent parameters. The first example considers the Hull-White model as a special case of generalised Black-Karasinski models [19] where the interest rate $r(t)$ is given by $r(t) = r(0)\,(1 + \nu X(t))^{1/\nu}$ with the parameter ν satisfying $0 < \nu \leq 1$ and where the process $X(t)$ follows the Ornstein-Uhlenbeck process

$$dX(t) = (\theta(t) - \kappa(t)X(t))\,dt + \sigma(t)dW(t), \; X(0) = 0.$$

The case $\nu = 1$ corresponds to the Hull-White model and we describe the pricing pde for this case. If $\bar{X}(t) = \mathbb{E}\,(X(t)|X(0) = 0)$, then it can be shown that

$$\bar{X}(t) = \lambda(t)\int_0^t \frac{\theta(s)}{\lambda(s)}\,ds, \quad \lambda(t) = e^{-\int_0^t \kappa(s)\,ds}.$$

With the process $Y(t)$ given by $X(t) = \bar{X}(t) + \lambda(t)Y(t)$, and $r(t, y) = r_0\left(1 + \bar{X}(t) + \lambda(t)y\right)$, the zero-coupon bond price $Z(t, y, T)$ at time t given that $Y(t) = y$ is the solution of the pde

$$\frac{\partial Z}{\partial t} + \frac{1}{2}\frac{\sigma^2(t)}{\lambda^2(t)}\frac{\partial^2 Z}{\partial y^2} - r(t, y)Z = 0, \quad (t, y) \in [0, T] \times \mathbb{R}, \tag{29}$$

with terminal condition $Z(T, y, T) = 1$.

To describe Jain's scheme for calculating the bond price $Z(t, y, T)$, we make the change of variable $\tilde{\tau} = T - t$ which results in the equation

$$\frac{\partial^2 Z}{\partial y^2} = f\left(y, \tilde{\tau}, Z, \frac{\partial Z}{\partial \tilde{\tau}}\right) = 2\lambda^2(\tilde{\tau})/\sigma^2(\tilde{\tau})\left(r(\tilde{\tau}, y)Z + \frac{\partial Z}{\partial \tilde{\tau}}\right). \tag{30}$$

For $0 \leq m \leq M$ and $0 \leq n \leq N$, consider uniformly spaced grid points $\left(y_m = mh, \tilde{\tau}_n = nk\right)$ in the $(y, \tilde{\tau})$-space. Also let $G(\tilde{\tau})$ be the function such that $1/G(\tilde{\tau}) = 2\lambda^2(\tilde{\tau})/\sigma^2(\tilde{\tau})$ and denote the value of $r\left(\tilde{\tau}_{n+\frac{1}{2}}, y_{m+q}\right)$ by $r^{n+\frac{1}{2}}_{m+q}$. If $\mu = k/h^2$, using (14) and the approximations in (15), we obtain

$$\begin{aligned} &\mu G\left(\tilde{\tau}_{n+\frac{1}{2}}\right)\left[Z^{n+\frac{1}{2}}_{m+1} - 2Z^{n+\frac{1}{2}}_m + Z^{n+\frac{1}{2}}_{m-1}\right] \\ &\quad = \frac{1}{12}\sum_{q=-1}^{1}\psi_q\left[\left(\zeta^{n+\frac{1}{2}}_{m+q} - 1\right)Z^{n+1}_{m+q} + \left(\zeta^{n+\frac{1}{2}}_{m+q} + 1\right)Z^n_{m+q}\right], \end{aligned} \tag{31}$$

where for $q = 0, \pm 1$, $\zeta^{n+\frac{1}{2}}_{m+q} = kr^{n+\frac{1}{2}}_{m+q}/2$ and $\psi_{\pm 1} = 1$ and $\psi_0 = 10$. It then follows that the resulting finite difference discretization is given by

$$\sum_{q=-1}^{1} b^{n+\frac{1}{2}}_{m+q}Z^{n+1}_{m+q} = \sum_{q=-1}^{1} c^{n+\frac{1}{2}}_{m+q}Z^n_{m+q}, \tag{32}$$

where the coefficients $b^{n+\frac{1}{2}}_{m+q}$ and $c^{n+\frac{1}{2}}_{m+q}$ for $q = \pm 1$ are given by

$$\begin{aligned} b^{n+\frac{1}{2}}_{m+q} &= \frac{1}{6}\left(\zeta^{n+\frac{1}{2}}_{m+q} + 1\right) - \mu G^{n+\frac{1}{2}}, \\ c^{n+\frac{1}{2}}_{m+q} &= \mu G^{n+\frac{1}{2}} - \frac{1}{6}\left(\zeta^{n+\frac{1}{2}}_{m+q} - 1\right), \end{aligned}$$

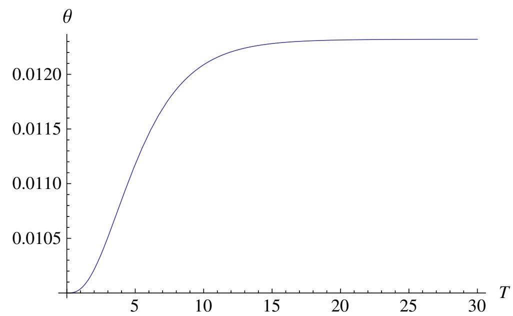

Figure 1 Plot of the calibrated mean-reversion level $\theta(t)$ under Hull-White model. This figure shows a plot of the mean reversion level θ under the Hull-White model.

and

$$b_m^{n+\frac{1}{2}} = \frac{5}{3}\left(\zeta_m^{n+\frac{1}{2}} + 1\right) + 2\mu G^{n+\frac{1}{2}},$$
$$c_m^{n+\frac{1}{2}} = -\left(2\mu G^{n+\frac{1}{2}} + \frac{5}{3}\left(\zeta_m^{n+\frac{1}{2}} - 1\right)\right).$$

Calibration to an initial term structure

Another example we consider concerns the calibration of the Hull-White model to an initial term structure. Fitting an observed initial term structure has usually been carried out using tree-based methods which can be viewed as explicit finite differences. Uhrig and Walter [20] developed a method based on an implicit finite difference method and Vetzal [21] suggested a method in combination with a fully implicit discretisation where the initial interest rate model is augmented by an additional interest rate process which is a function only of time. A more recent method in the context of the Crank-Nicolson scheme has been described by Daglish [22].

Although it is possible to develop a similar technique in the context of Jain's scheme, we have opted for a procedure described in Hull and White [23] where the initial term structure is assumed to follow the CIR model and then calibrate the Hull-White model to the CIR bond prices. This allows an analytical expression for the mean reversion level $\theta(t)$ which for the Hull-White model is given by

$$\theta(t) = \frac{\partial f}{\partial t}(0, t) + \kappa f(0, t) + \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right),$$

where $f(0, t)$ is the forward rate given by

$$f(0, t) = \frac{\partial}{\partial t}\left(\ln A(0, t)\right) - r\frac{\partial}{\partial t}\left(B(0, t)\right)$$

and $A(0, t)$ and $B(0, t)$ are expressions which can be obtained from the $A(t, T)$ and $B(t, T)$ terms in the CIR analytical bond price.

Using the parameters $\kappa = 0.2$, $r = 0.05$, $\theta = 0.05$ and $\sigma = 0.2$, a plot of the mean reversion level $\theta(t)$ is shown in Figure 1.

Results and discussion

We describe several numerical examples carried out to illustrate the good performance of the new technique for pricing stock options, bonds, European, American and Bermudan bond options and comparisons are drawn with the numerical results obtained by the Crank-Nicolson (CN) scheme. In the tables, we have also indicated the value of the parabolic mesh ratio μ used for Jain's scheme and the parameter used for the different models, and the exact values where they exist. In all our examples, we choose $r_{\min} = 0$ and $r_{\max} = 0.5$ unless specified otherwise. All computations have been performed using Mathematica 7.0 on a Core i5 laptop with 4GB RAM and speed 4.60 GHz.

European stock options

Our first numerical example considers the pricing of European call options under the Black-Scholes model using the numerical scheme (16). Table 1 shows computed prices, errors (difference between exact and computed prices), convergence rates and cpu timings. We observe that Jain's scheme with a local mesh refinement yields numerical solutions which exhibit the expected fourth-order convergence rate. For this example, Jain's scheme

Table 1 European options under the Black-Scholes model

$T = 1$, $S = 100$, $K = 100$, $\sigma = 0.3$, $r_0 = 0.1$, $\delta = 0.06$, $x_{\min} = -1$, $x_{\max} = 1$, $\mu = 0.5$								
	CN				Jain's scheme			
M	Price	Error	Order	cpu(s)	Price	Error	Order	cpu(s)
2^5	12.895545	5.7(-02)	-	0.003	12.943103	9.2(-03)	-	0.006
2^6	12.938229	1.4(-02)	2.004	0.005	12.951777	5.6(-04)	4.043	0.014
2^7	12.948815	3.5(-03)	2.002	0.013	12.952305	3.2(-05)	4.114	0.041
2^8	12.951457	8.8(-04)	2.001	0.030	12.952335	1.3(-06)	4.542	0.204
2^9	12.952117	2.2(-04)	2.000	0.081	12.952337	7.1(-08)	4.283	1.310
Exact=12.952337								

Table 2 Bond prices under the CIR model for $T = 5$

$\gamma = 1/2$, $T = 5$, $\kappa = 0.5$, $\theta = 0.08$, $\sigma = 0.1$, $\mu = 500$, $r_0 = 0.05$, $r_{\min} = 0$, $r_{\max} = 0.5$								
	CN				Jain's scheme			
M	**Price**	**Error**	**Order**	**cpu(s)**	**Price**	**Error**	**Order**	**cpu(s)**
10	71.035945	2.0(-03)	-	0.002	71.078676	4.1(-02)	-	0.002
20	71.037683	2.5(-04)	2.972	0.003	71.039171	1.2(-03)	5.046	0.003
40	71.037894	4.3(-05)	2.545	0.006	71.037998	6.1(-05)	4.349	0.056
80	71.037927	1.1(-05)	1.950	0.013	71.037941	3.6(-06)	4.072	0.045
160	71.037935	2.8(-06)	2.000	0.044	71.037938	2.3(-07)	3.998	0.217
320	71.037937	7.0(-07)	2.000	0.145	71.037938	1.4(-08)	3.997	1.248
Exact	71.037938							

is superior to the Crank-Nicolson (CN) scheme in terms of accuracy of computed solutions. For illustration, Jain's scheme yields a solution with an error of 5.6×10^{-4} in 14 milliseconds (ms) whereas for approximately the same computational time, the CN computed solution has an accuracy level of 3.5×10^{-3}.

Zero-coupon bonds

We now compare the performances of CN and Jain's scheme for computing the price of a zero-coupon bond under the CIR model for which an analytical solution exists and is given by (4). Numerical results for the CIR model are shown in Table 2 when the bond has a maturity of $T = 5$ years and the face value is $\check{f} = 100$. Although the Crank-Nicolson scheme is faster in the sense of taking a lower computational time for comparable accuracy levels (44 ms for an error of 2.8×10^{-6} for CN v/s 45 ms for Jain for an error of 3.6×10^{-6}), we observe that fourth-order convergence rates are achieved with Jain's scheme. Using only 160 spatial grid points, the high-order scheme

Table 3 CIR and Vasicek bond prices for $T = 30$

$T = 30$, $\kappa = 0.5$, $\theta = 0.08$, $\sigma = 0.1$, $\mu = 500$, $r_0 = 0.05$, $r_{\max} = 0.5$								
	Jain's Scheme							
	CIR				Vasicek			
	$r_{\min} = 0$				$r_{\min} = -0.5$			
M	**Price**	**Error**	**Order**	**cpu(s)**	**Price**	**Error**	**Order**	**cpu(s)**
10	10.038708	7.4(-03)	-	0.003	16.270460	2.6(-01)	-	0.002
20	10.030512	7.5(-04)	3.314	0.008	16.514895	1.5(-02)	4.112	0.005
40	10.031187	7.4(-05)	3.332	0.039	16.589528	9.4(-04)	4.002	0.011
80	10.031256	4.9(-06)	3.911	0.231	16.529830	5.8(-05)	4.000	0.063
160	10.031261	3.1(-07)	4.001	0.985	16.529885	3.7(-06)	3.999	0.394
320	10.031261	1.9(-08)	4.002	2.013	16.529889	2.3(-07)	3.988	1.892
Exact	10.031261				16.529889			

Table 4 Bond prices under CKLS model for different values of the parameter γ

$T = 5$, $\kappa = 0.5$, $\theta = 0.08$, $r_0 = 0.08$, $\sigma = 0.1$, $\mu = 500$				
	γ			
M	**0.4**	**0.6**	**0.8**	**1.0**
10	71.270936	71.056493	70.973343	71.902727
20	71.188097	70.955286	70.870295	70.947619
40	71.184381	70.951005	70.869050	70.847016
80	71.184204	70.950756	70.869206	70.841470
160	71.184195	70.950741	70.869194	70.841438
CN	71.184195	70.950737	70.869191	70.841408

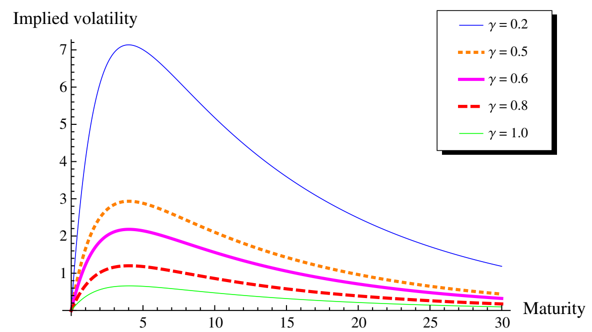

Figure 2 Implied volatility curve under the CKLS model. This figure shows the bond price volatilities for different values of the CKLS parameter γ.

yields a numerical solution that agrees to eight-digits with the analytical price.

Results for a longer maturity ($T = 30$) bond pricing problem under the Vasicek and CIR models are given in Table 3. For the CIR model, the scheme is able to compute a numerical solution which agrees to five-digits with the exact price in only 6 milliseconds for CIR model. For other values of the CKLS parameter γ for which no closed form expressions are known, numerical results are given in Table 4. The last row in this Table gives the corresponding bond prices computed by the Crank-Nicolson scheme using 160 grid points. The high accuracy of Jain's scheme obtained for known analytical results brings us to conclude that CN scheme becomes less accurate with increasing γ.

We show in Figure 2, the bond price volatilities for different values of the parameter γ. It is seen that the volatility curves become flatter with increasing values of γ.

European zero-coupon bond options

We consider the pricing of European bond options under the CIR model for which an analytical solution is given by (5). In Table 5, we give the prices, errors, convergence rates and cpu timings for European call options for different maturities, T_o, on bonds with maturity $T = 10$. For this bond option problem, the computed numerical solutions exhibit fourth-order convergence. For the short maturity option, Jain's scheme requires only 5 milliseconds to compute a numerical solution which agrees with the exact solution in the first five digits and for the longer maturity bond option, five-figure accuracy is reached in 91 milliseconds. These results clearly demonstrate the good performance of Jain's scheme for pricing European bond options. In Table 6, we give computed prices for values of the parameter γ for which no analytical solutions exist.

American zero-coupon bond options

We next consider the pricing of American bond options and we compare the results obtained using Jain's scheme with those computed by CN. Table 7 shows the results under the CIR model for pricing an American put option with strike $K = 60$, bond maturity $T_o = 5$ years and bond maturity $T = 10$ years. The difference ($|\text{Value}_M - \text{Value}_{2M}|$) between the computed prices on grids with M and $2M$ points and the ratio ($\text{error}_M/\text{error}_{2M}$) are shown. We observe that the differences between the numerical solutions obtained by Jain's scheme decrease by a factor of approximately 16 whereas the differences for the CN computed solutions decrease by a factor of approximately 4. In Table 8, corresponding American option prices for different values of the CKLS parameter $\gamma = 0.3$, $\gamma = 0.4$ and $\gamma = 0.6$ are given. We observe that the numerical solutions converge and the American option price is higher with increasing γ.

Coupon bonds

We next consider the pricing of coupon bonds with regular coupon payments. Table 9 gives the prices, errors, convergence rates and cpu timings for a bond with maturity $T = 5$ and 5% coupon paid annually under the CIR

Table 5 European bond option prices under the CIR model for option maturities $T_o = 2$ and $T_0 = 5$

$\gamma = 1/2$, $T = 10$, $K = 35$, $\kappa = 0.5$, $\theta = 0.08$, $\sigma = 0.1$, $\mu = 1000$, $r_0 = 0.08$, $r_{max} = 0.5$								
	Bond options - Jain's Scheme							
	$T_o = 5$				$T_o = 2$			
M	Price	Error	Order	cpu(s)	Price	Error	Order	cpu(s)
10	21.823819	5.6(-02)	-	0.001	15.549299	3.7(-02)	-	0.001
20	21.875472	4.7(-03)	3.578	0.003	15.580977	5.3(-03)	2.796	0.002
40	21.879880	3.1(-04)	3.911	0.005	15.585949	3.6(-04)	3.903	0.003
80	21.880174	1.9(-05)	3.985	0.091	15.586282	2.3(-05)	3.995	0.005
160	21.880192	1.2(-06)	4.000	0.296	15.586304	1.4(-06)	3.999	0.281
320	21.880193	7.7(-08)	4.001	1.997	15.586305	9.1(-08)	3.998	1.981
Exact	21.880193				15.586305			

Table 6 European call option prices under CKLS for different values of the parameter γ

$T = 10,\ T_o = 5,\ \kappa = 0.5,\ \theta = 0.08,\ r_0 = 0.08,\ \sigma = 0.1,\ \mu = 1000$				
	γ			
M	0.4	0.6	0.8	1.0
10	22.083910	21.654108	21.479889	21.163772
20	22.130553	21.716483	21.559178	21.459844
40	22.134778	21.721184	21.563424	21.499361
80	22.135111	21.721490	21.563760	21.504963
160	22.135135	21.721510	21.563779	21.505273
CN	22.135125	21.721496	21.563767	21.505279

Table 7 American put option prices under the CIR model

$T = 10,\ T_o = 5,\ K = 60,\ \kappa = 0.5,\ \theta = 0.08,\ \sigma = 0.1,\ \mu = 1000,\ r_0 = 0.08$						
	American put					
	CN			Jain's scheme		
M	Price	Error	Ratio	Price	Error	Ratio
10	14.574079	-	-	14.579502	-	-
20	14.573020	1.1(-03)	-	14.573140	5.7(-03)	-
40	14.572778	2.4(-04)	4.3774	14.572723	3.9(-04)	15.276
80	14.572715	6.2(-05)	3.8743	14.572696	2.5(-05)	15.377
160	14.572670	1.6(-05)	4.0265	14.572694	1.6(-06)	15.882

Table 8 American put option prices under CKLS for different values of γ

$T = 10,\ T_o = 5,\ K = 60,\ \kappa = 0.5,\ \theta = 0.08,\ \sigma = 0.1,\ r_0 = 0.08$			
M	$\gamma = 0.3$	$\gamma = 0.4$	$\gamma = 0.6$
10	13.789399	13.983794	14.771959
20	13.776068	14.263980	14.765391
40	13.773518	14.263511	14.764967
80	13.772914	14.263477	14.764940
160	13.772751	14.263475	14.764940

Table 9 Coupon bond prices under the CIR model with an annual coupon of 5% and face value 100

$T = 5,\ \kappa = 0.5,\ \theta = 0.08,\ \sigma = 0.1,\ r_0 = 0.05,\ \mu = 500$								
	CN				Jain's scheme			
M	Price	Error	Order	cpu(s)	Price	Error	Order	cpu(s)
10	91.698985	3.3(-03)	-	0.002	91.700726	1.4(-03)	-	0.003
20	91.699504	1.8(-04)	1.310	0.003	91.699393	6.7(-05)	4.334	0.011
40	91.699340	6.7(-05)	1.433	0.006	91.699328	4.2(-06)	4.069	0.049
80	91.699340	1.6(-05)	2.038	0.014	91.699324	2.6(-07)	3.998	0.292
160	91.699327	4.1(-06)	2.001	0.053	91.699323	1.6(-08)	3.999	1.305
Exact=91.699323								

Table 10 European option for CIR model on bond with face value 100 with an annual coupon of 10% compounded semiannually

$T = 5,\ T_o = 1.5,\ K = 100,\ \kappa = 0.2,\ \theta = 0.1,\ \sigma = 0.06,\ r_0 = 0.1,\ \mu = 500$						
	CN			Jain		
M	Price	Error	Order	Price	Error	Order
20	1.361296	3.2(-02)	-	1.405948	7.6(-02)	-
40	1.315291	1.4(-02)	1.129	1.313588	1.4(-02)	2.239
80	1.326963	2.8(-03)	2.383	1.326753	2.9(-03)	2.439
160	1.329623	1.1(-04)	2.698	1.329334	3.9(-04)	2.910
320	1.329768	3.8(-05)	1.480	1.329749	1.9(-05)	4.359
Exact=1.329730						

Table 11 European option for CIR model on bond with face value 100 and an annual coupon of 10% compounded semiannually at different strike prices

$\kappa = 0.2,\ \theta = 0.1,\ \sigma = 0.06,\ r_0 = 0.1,\ \mu = 400$					
	Strike price (K)				
T_o	95.0	97.5	100	102.5	105.0
0.5	4.2979 (4.2978)	2.3219 (2.3216)	0.9373 (0.9373)	0.2525 (0.2525)	0.0402 (0.0403)
1.0	4.3189 (4.3192)	2.5430 (2.5430)	1.2349 (1.2348)	0.4642 (0.4642)	0.1256 (0.1256)
1.5	4.2513 (4.2514)	2.5853 (2.5852)	1.3298 (1.3297)	0.5474 (0.5477)	0.1691 (0.1691)
2.0	4.1161 (4.1161)	2.5214 (2.5214)	1.3080 (1.3080)	0.5420 (0.5420)	0.1665 (0.1665)
3.0	3.7288 (3.7288)	2.2043 (2.2044)	1.0471 (1.0471)	0.3591 (0.3593)	0.0760 (0.0762)
4.0	3.3231 (3.3231)	1.7691 (1.7691)	0.5944 (0.5944)	0.0779 (0.0779)	0.0013 (0.0013)

The exact option price is given in brackets.

Table 12 American call option with strike price $K = 100$ under CIR model on a bond with an annual coupon of 10% compounded semiannually

$T = 10,\ T_o = 5,\ \kappa = 0.1,\ \theta = 0.08,\ \sigma = 0.1,\ r_0 = 0.05,\ \mu = 1000$					
	American option				European option
M	Price	Error	Order	cpu(s)	Price
2^5	17.290119	-	-	0.006	8.525995
2^6	17.283180	6.9(-03)	-	0.051	8.524684
2^7	17.282677	5.0(-04)	3.785	0.152	8.524020
2^8	17.282647	3.0(-05)	4.062	0.286	8.523980
2^9	17.282644	2.3(-06)	3.667	1.992	8.523920
Tree	17.7237				8.5257
Exact	-				8.523929

Table 13 Bermudan put option under CIR model on a bond with face value 1000 with coupon of 4% compounded annually

$T = 5,\ T_o = 3,\ K = 800,\ \kappa = 0.1,\ \theta = 0.08,\ \sigma = 0.075,\ r_0 = 0.05,\ \mu = 1000$						
	CN			Jain's scheme		
M	Price	Error	Order	Price	Error	Order
2^7	0.022161	-	-	0.019695	-	-
2^8	0.021101	7.7(-03)	-	0.020655	9.6(-04)	-
2^9	0.020814	1.1(-03)	1.887	0.020716	6.1(-05)	3.976
2^{10}	0.020745	2.9(-04)	2.059	0.020720	3.6(-06)	4.083

Table 14 Zero-coupon bond prices with face value 1 under the Hull-White model for $T = 1$ and $T = 10$

$\theta = 0.1$, $\kappa = 0.2$, $\sigma = 0.1$, $r_0 = 0.02$, $\mu = 10$, $y_{\min} = -1$, $y_{\max} = 1$						
	$T = 1$			$T = 10$		
M	**Price**	**Error**	**Order**	**Bond Price**	**Error**	**Order**
10	0.97928152	1.5(-07)	-	0.77369533	5.6(-07)	-
20	0.97928166	9.5(-08)	3.999	0.77369586	3.5(-08)	4.000
40	0.97928167	5.9(-10)	4.000	0.77369589	2.2(-09)	3.999
80	0.97928167	3.7(-11)	4.000	0.77369589	1.4(-10)	4.000
	Exact=0.97928167			Exact=0.77369589		

Table 15 Zero-coupon bond prices with face value 1 for the Hull-White model when $T = 20$ and $T = 30$

$\theta = 0.1$, $\kappa = 0.2$, $\sigma = 0.1$, $r_0 = 0.02$, $\mu = 10$, $y_{\min} = -1$, $y_{\max} = 1$						
	$T = 20$			$T = 30$		
M	**Price**	**Error**	**Order**	**Price**	**Error**	**Order**
10	0.57678908	4.2(-07)	-	0.42784324	3.1(-07)	-
20	0.57678730	2.6(-08)	3.999	0.42784352	1.9(-08)	4.000
40	0.57678732	1.6(-09)	4.000	0.42784354	1.2(-09)	4.000
80	0.57678733	1.0(-10)	4.001	0.42784355	7.6(-11)	4.000
	Exact=0.57678733			Exact=0.42784355		

model. Comparison with the Crank-Nicolson scheme and the exact solution computed using the analytical formula in (6) shows that Jain's scheme yields the expected fourth order convergence rate.

In Table 10, we compare the CN and Jain's scheme to price a European call option with maturity $T_o = 1.5$ at strike $K = 100$ on a coupon bearing bond with maturity $T = 10$ years with a coupon of 10% compounded semi-annually under the CIR term structure model. The exact price is computed using the analytical formula provided in (7). Numerical results and analytical prices when we vary the strike price K and the time-to-maturity T_o of the option are given in Table 11. We observe that prices are accurately computed.

In our next example, we consider the pricing of an American call option with a maturity of $T_o = 5$ years on a coupon bond with a maturity of 10 years when the underlying interest rate model is CIR. Computed American

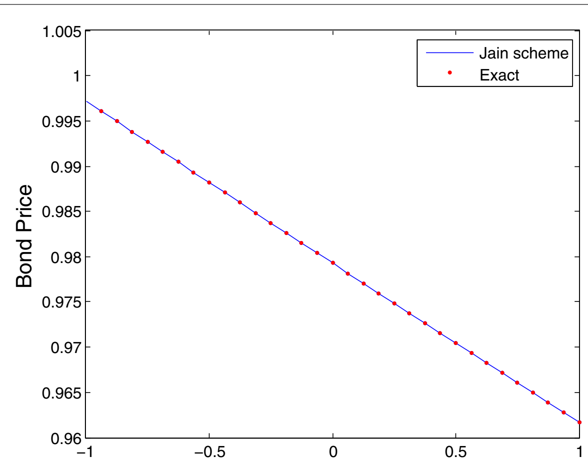

Figure 3 Bond price curve under Hull-White model for maturity $T = 1$ for time dependent coefficients. This figure shows the computed and the analytical bond prices under the Hull-White model when $T = 1$ using the parameters in Table 14.

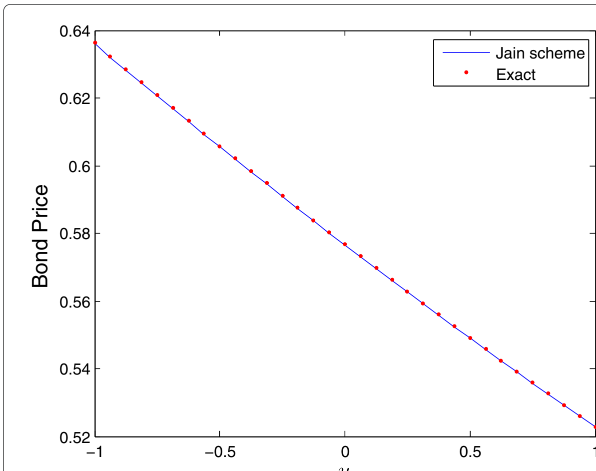

Figure 4 Bond price curve under Hull-White model for maturity $T = 20$ for time dependent coefficients. This figure shows the computed and the analytical bond prices under the Hull-White model when $T = 20$ using the parameters in Table 15.

Table 16 Bond prices with face value 1 under Hull-White model for $T = 1$ and $T = 2$ using calibrated $\theta(t)$

$\kappa = 0.2$, $\sigma = 0.2$, $r_0 = 0.05$, $\mu = 10$, $y_{\min} = -1$, $y_{\max} = 1$						
	$T = 1$			$T = 2$		
M	Price	Error	Order	Price	Error	Order
10	0.95100616	-	-	0.90404652	-	-
20	0.95100752	1.4(-06)	-	0.90405037	3.8(-06)	-
40	0.95100765	1.3(-07)	3.432	0.90405061	2.4(-07)	4.005
80	0.95100766	7.8(-09)	4.004	0.90405062	1.5(-08)	4.022
160	0.95100766	4.8(-10)	4.016	0.90405062	8.7(-10)	4.090
	Exact=0.95150115			Exact=0.90659982		

option prices and corresponding European prices are given in Table 12. For this problem, the tree method in [24] gives a price of 17.7237. Our computations show that Jain's scheme converges to a price of 17.28264. Given the good properties of our scheme in other test examples, we conclude that the tree method does not give a good approximation of the true American option price.

In Table 13, we give the numerical results when we price a Bermudan put option with a maturity of 3 years on a 5-year 4% annual coupon bond under the CIR model and where early exercise is only possible at coupon dates. We see that Jain's scheme yields fourth order convergence rates for this problem also.

Time dependent parameters

The next example shows that Jain's scheme is stable and efficient for problems with time dependent parameters. We compute the zero-coupon bond price described by the pde (29). Tables 14 and 15 show computed prices for zero-coupon bonds with maturities of $T = 1$, 10, 20 and 30 years respectively. For this time-dependent coefficient problem, we find that Jain's scheme yields uniform fourth-order convergence rates. No oscillations are observed in computed bond prices. This is shown in Figure 3 and Figure 4 which shows the bond price curves for bonds with maturity 1 and 20 years respectively.

Our final example concerns the fitting of the Hull-White model to an initial term structure using the time-dependent mean reversion level $\theta(t)$ shown in Figure 1. The computed bond prices by Jain's scheme shown in Table 16 are close to the CIR analytical bond prices. We can therefore conclude from the results in Tables 15 and 16 that Jain's has good stability properties with financial problems with time-dependent parameters.

Conclusions

The Crank-Nicolson scheme is widely used for pricing options on stocks and bonds. For the Black-Scholes equation, the scheme produces oscillations in the hedging parameters such as the stock option's delta and gamma. The solution to overcome this problem is the Rannacher time stepping [25]. This work described a high-order scheme for solving the Black-Scholes equation and an analysis was carried out. We demonstrated that computed option prices had the expected fourth-order convergence. We then extended the numerical scheme for pricing a wide variety of interest rate derivatives. Numerical examples were conducted to demonstrate the good properties of the scheme and comparisons were drawn against the Crank-Nicolson discretisation. The numerical scheme was then extended to solve the American option problem on coupon bonds and the scheme's ability to compute highly accurate convergent approximations was demonstrated numerically. It was shown that scheme is also stable for pricing financial problems described by pricing equations with time dependent coefficients. In summary, the scheme proposed in this paper is a more accurate technique than the Crank-Nicolson scheme and has comparable and at times superior efficiency in terms of computational speed.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

NT carried out all numerical tests and contributed to the mathematical derivations and writing of the paper. YT and MB contributed to the mathematical derivations and writing of the paper. All authors read and approved the final manuscript.

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References

1. Black, F, Scholes, M: The pricing of options and other corporate liabilities. J. Pol Econ. **81**(3), 637–654 (1973)
2. Tangman, DY, Gopaul, A, Bhuruth, M: Numerical pricing of options using high-order compact finite difference schemes. J. Comp. Appl. Math. **218**, 270–280 (2008)
3. Jain, MK, Jain, RK, Mohanty, RK: A fourth order difference method for the one dimensional general quasilinear parabolic differential equation. Numer. Methods. Partial Differential Eq. **6**, 311–319 (1990)

4. Rigal, A: High order difference schemes for unsteady one-dimensional diffusion-convection problems. J. Comp. Phys. **114**, 59–76 (1994)
5. Vasicek, O: An equilibrium characterization of the term structure. J. Financ. Econ. **5**, 177–188 (1977)
6. Cox, JC, Ingersoll, JE, Ross, SA: A theory of the term structure of interest rates. Econometrica. **53**, 385–407 (1985)
7. Sorwar, G, Barone-Adesi, G, Allegretto, W: Valuation of derivatives based on single-factor interest rate models. Glob Finance J. **18**, 251–269 (2007)
8. Tangman, DY, Thakoor, N, Dookhitram, K, Bhuruth, M: Fast approximations of bond option prices under CKLS models. Finance Res. Lett. **8**, 206–212 (2011)
9. Chesney, M, Elliott, RJ, Gibson, R: Analytic solutions for the pricing of American bond and yield options. Math. Finance. **3**, 277–294 (1993)
10. Tian, Y: A simplified binomial approach to the pricing of interest rate contingent claims. J. Fin. Engrg. **1**, 19–37 (1992)
11. Allegretto, W, Lin, Y, Yang, H: Numerical pricing of American put options on zero-coupon bonds. Appl. Numer. Math. **46**, 113–134 (2003)
12. Chan, KC, Karolyi, GA, Longstaff, FA, Sanders, AB: An empirical comparison of alternative models of the short-term interest rate. J. Finance. **1**, 1209–1227 (1992)
13. Wong, HY, Zhao, J: An artificial boundary method for American option pricing under the CEV model. SIAM J. Numer. Anal. **46**, 2183–2209 (2008)
14. Jamshidian, F: An exact bond option formula. J. Finance. **44**, 204–210 (1989)
15. Longstaff, FA: The valuation of options on coupon bonds. J. Bank Finance. **17**, 27–42 (1993)
16. Bensoussan, A: On the theory of option pricing. Acta Appl. Math. **2**, 139–158 (1984)
17. Zhang, J, Sun, H: High-order compact scheme with multigrid local mesh refinement procedure for convection-diffusion problems. Comput. Methods Appl. Mechanics Engrg. **191**, 4861–4874 (2002)
18. Ikonen, S, Toivanen, J: Operator splitting methods for American option pricing. Appl. Math. Lett. **17**, 809–814 (2004)
19. Tourrucôo, F, Hagan, PS, Schleiniger, GF: Approximate formulas for zero-coupon bonds. Appl. Math. Finance. **14**(3), 207–226 (2007)
20. Uhrig, M, Walter, U: A new numerical approach for fitting the initial yield curve. J. Fixed Income. **5**, 82–90 (1996)
21. Vetzal, KR: An improved finite difference approach to fitting the initial term structure. J. Fixed Income. **7**, 62–81 (1998)
22. Daglish, T: Lattice methods for no-arbitrage pricing of interest rate securities. J. Derivatives. **18**, 7–19 (2010)
23. Hull, J, White, A: Pricing interest-rate derivative securities. Rev. Financ. Stud. **3**, 573–392 (1990)
24. Nawalkha, SK, Beliaeva, NA, Soto, GM: Dynamic Term Structure Modeling: The Fixed Income Valuation Course. Wiley Finance, John Wiley and Sons, Hoboken (2007)
25. Giles, M, Carter, R: Convergence analysis of Crank-Nicolson and Rannacher time-marching. J. Comput. Finance. **94**, 89–112 (2006)

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